Pergamon

Int. J. Solids Structures Vol. 31, No. 18, pp. 2433–2444, 1994

0020–7683/94 $7.00 + .00

0020–7683(94)E0054–Y

AN ANALYTICAL METHOD FOR SHAKEDOWN PROBLEMS WITH LINEAR KINEMATIC HARDENING MATERIALS

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(*Received* 11 *March* 1993; *in revised form* 25 *March* 1994)

Abstract—This paper deals with shakedown problems of systems with linear kinematic hardening material. Based on the fact that a system with linear kinematic hardening material can only fail locally in the form of alternating plasticity, an analytical method for determining the shakedown load factors was developed. First this method is derived in a general form and then applied to some examples. Finally, the analytical results are compared with those obtained by other analytical and numerical methods.

1. INTRODUCTION

Melan (1938a,b) formulated the static shakedown theorems for both perfectly plastic and linear, unlimited kinematic hardening materials. Koiter (1956) introduced a kinematic shakedown theorem for perfectly plastic materials which can be regarded as a dual one of Melan's static theorem. Since then shakedown theorems dealing with temperature loading, dynamic loading and geometrically nonlinear effects etc. have been elaborated by different authors (Prager, 1956; Rozenblum, 1957; Ceradini, 1969; König, 1969; Maier, 1972; Corradi and Maier, 1973; Weichert, 1986). The shakedown theorem for material with nonlinear kinematic hardening has been formulated for the first time by Neal (1950), who used the 1-D Masing (1924) overlay model for describing the nonlinear kinematic behaviour of materials. Neal's formulation is only valid for 1-D stress state problems. Recently Stein *et al.* (1992, 1993a,b) and Stein and Zhang (1992) used a 3-D overlay model to describe the nonlinear hardening behaviour of materials. They also formulated the corresponding shakedown theorem. Following that method, one can determine the failure mechanism of a system under cyclic loadings. For instance, a system consisting of linear kinematic hardening material fails only by alternating plasticity [see Stein *et al.* (1993)].

One purpose of shakedown investigations for a system is to find a maximum factor, by which a given convex load domain is allowed to extend on the condition that the system still shakes down, i.e. that the total plastic energy becomes stationary. In the context of computational mechanics such a shakedown problem can be treated as an optimization problem by using FEM. Generally, the dimension of the optimization problem via FEM is very large; Mahnken (1992) and Zhang (1992) have used special numerical methods, namely a dual method and a reduced base technique, to solve these problems. To avoid these extensive computations, direct methods for determination of the shakedown load factors were developed. Rozenblum (1958) suggested an approximate method, by which some suitable field of residual stresses should be chosen. In general, this is not easy to do, especially for systems with complicated geometries and boundary conditions. By using the property that only alternating plasticity is relevant in the case of unlimited kinematic hardening, Zarka *et al.* (1978) introduced a straightforward method for determining the steady state of stresses and strains of a system, provided that the system shakes down, but the shakedown as an optimization problem is not treated there.

In this work, the difference of the residual stress and the backstress is defined as the effective residual stress, following Zarka *et al.* (1978). The property that a system with linear kinematic hardening material only fails by alternating plasticity, has been used. Thus, a shakedown problem can be solved locally for some point in the system. For solving this local optimization problem, an analytical method is developed. The advantage of this method is that only the maximum effective elastic stress of the system has to be calculated, and the shakedown load factor follows from an analytical form. In applying this method, several typical shakedown problems are studied. The results are compared with those obtained by using analytical and numerical methods.

2. SHAKEDOWN WITH LINEAR KINEMATIC HARDENING

For a system with linear kinematic hardening material, the shakedown problem can be formulated as follows [see Stein (1993a)]:

$$\beta \rightarrow \max. \tag{1}$$

$$\Phi(\beta \boldsymbol{\sigma}^{\mathrm{E}}(j)+\mathbf{y}) \leqslant \sigma_0^2 \quad \forall j \in \mathscr{J}, \tag{2}$$

where β is the shakedown load factor which is to be found, Φ the von Mises yield function, $\boldsymbol{\sigma}^{\mathrm{E}}$ the elastic stress vector, σ_0 the initial yield stress and $\mathscr{J}$ the set of all load vertices. The effective residual stress $\mathbf{y}$ is defined by (Zarka *et al.*, 1978; Jiang and Leckie, 1992)

$$\mathbf{y} = \boldsymbol{\rho} - \boldsymbol{\alpha}. \tag{3}$$

The optimization problem (1)–(2) has a very simple structure. It has been shown by Stein *et al.* (1993a) that the maximum value of the load factor β_s is determined by

$$\beta_s = \min_{i \in \mathscr{I}} \bar{\beta}_i, \tag{4}$$

with $\bar{\beta}_i$ being the solution of subproblem for the point $\mathbf{x}_i$

$$\beta_i \rightarrow \max \tag{5}$$

$$\Phi(\beta_i \boldsymbol{\sigma}_i^{\mathrm{E}}(j)+\mathbf{y}_i) \leqslant \sigma_0^2 \quad \forall j \in \mathscr{J}, \tag{6}$$

where $\mathscr{I}$ is the set of all points of the system.

The dimension of problem (5)–(6) is very low. The number of the unknowns is NSK + 1 with NSK being the number of stress components at point $\mathbf{x}_i$. For 3-D problems the number of unknowns is seven. The number of constraints is identical to the number of vertices of the load domain.

By identifying the vector $(-\mathbf{y}_i)$ with the shift of the initial yield surface in the stress space $\mathscr{S}$, the problem (5)–(6) can be described with the following geometrical interpretation: find the maximum affine enlargement of elastic domain $\mathscr{S}_i^{\mathrm{E}}$ and the corresponding shift $(-\mathbf{y}_i)$ of the yield surface on condition that the enlarged elastic domain $\beta_i \mathscr{S}_i^{\mathrm{E}}$ is still contained in the shifted yield surface.

In view of the above interpretation, eqn (4) implies that for systems consisting of an unlimited kinematic hardening material the shakedown load factor is determined by that point $\mathbf{x}_{i_p}$ in the system where the maximum enlargement β_{i_p} of the elastic domain $\mathscr{S}_{i_p}^{\mathrm{E}}$ is the smallest in comparison to all other points. Thus, $\beta_s = \beta_{i_p}$ holds. For a kinematic hardening material the size and shape of the yield surface remain unchanged during the yielding, and thus it is evident that the shakedown loads of systems consisting of unlimited kinematic hardening material cannot be infinite as expected for monotone loading. The only exception is that elastic domains $\mathscr{S}_i^{\mathrm{E}}$, for all points of the system, are straight lines and coincide with the diagonal in the space of principal stresses. This means that all elastic stresses $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x})$ are hydrostatic. However, this case is almost meaningless in practice.

3. ANALYTICAL DETERMINATION OF SHAKEDOWN LIMIT

According to the analysis in the last section the shakedown behaviour of a system with linear kinematic hardening material is dominated by some point in the system, where the maximum enlargement of the elastic domain is smallest in comparison to all other points. If the number of the constraints—which is equal to the number of vertices of the load domain—is small, it is possible to solve the optimization problem (5)–(6) analytically.

Let us consider a system in Fig. 1 subjected to the biaxial loadings p_1 and p_2 which may vary between zero and certain maximum magnitudes $\bar{p}_1$ and $\bar{p}_2$ independently. The maximum enlarging of the loading domain

$$0 \leqslant p_1 \leqslant \bar{p}_1,$$

$$0 \leqslant p_2 \leqslant \bar{p}_2$$

has to be found on the condition that the system will shake down. The restriction to four vertices of the load domain is caused by the complexity of the analytical solution. Furthermore, most practical applications need only four vertices.

Assuming that the maximum effective stress would appear at some point in the system, say point A, we need to solve the optimization problem

$$-\beta \to \min \tag{7}$$

$$\Phi[\beta\boldsymbol{\sigma}^{\mathrm{E}}(j)+\mathbf{y}]-\sigma_0^2 \leqslant 0, \quad j = 1,2,3,4 \tag{8}$$

only for point A. $\boldsymbol{\sigma}^{\mathrm{E}}(j)$ denotes the elastic stress at point A for load vertex j.

Usually the stress at a point has six components. The problem (7)–(8) has therefore seven unknowns β, y_{11}, y_{12}, y_{13}, y_{22}, y_{23} and y_{33}. If we work with principal stresses, $\boldsymbol{\sigma}^{\mathrm{E}}$ has only three components, and the number of unknowns of problem (7)–(8) remains four. For a plane stress problem it has only three unknowns. This reduction makes the problem much simpler to solve.

For the simplicity of notation, we let $\sigma_{\mathrm{I}}^{\mathrm{E}}(2)$ (the first principal stress at load vertex 2) be indicated by S_{12}, $\sigma_{\mathrm{I}}^{\mathrm{E}}(3)$ by S_{13}, $\sigma_{\mathrm{I}}^{\mathrm{E}}(4)$ by S_{14} and so on. Furthermore, we assume that the principal stress at load vertex 3 is a sum of the stresses from load vertex 2 and 4, i.e.

$$S_{13} = S_{12}+S_{14}, \quad S_{23} = S_{22}+S_{24}, \quad S_{33} = S_{32}+S_{34}. \tag{9}$$

Usually there are nonlinear relations between principal stresses, unlike eqn (9), but we will begin our analysis with the simplest case. Physically, it corresponds to a situation for which the principal stresses at all load vertices have the same directions.

By the above arrangement we can express the principal stresses for point A by

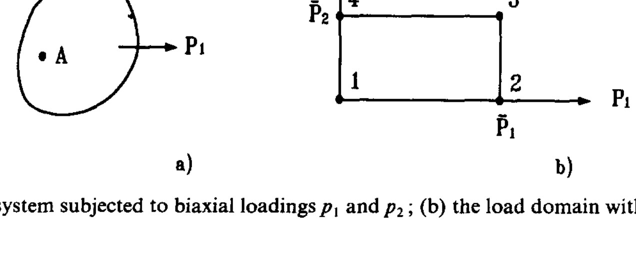

Fig. 1. (a) A system subjected to biaxial loadings p_1 and p_2; (b) the load domain with four vertices.

$$\sigma^E(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \sigma^E(2) = \begin{bmatrix} S_{12} \\ S_{22} \\ S_{32} \end{bmatrix},$$

$$\sigma^E(3) = \begin{bmatrix} S_{13} \\ S_{23} \\ S_{33} \end{bmatrix} = \begin{bmatrix} S_{12}+S_{14} \\ S_{22}+S_{24} \\ S_{32}+S_{34} \end{bmatrix}, \quad \sigma^E(4) = \begin{bmatrix} S_{14} \\ S_{24} \\ S_{34} \end{bmatrix}.$$

The solutions of problem (7)–(8) must satisfy the Kuhn–Tucker conditions, i.e.

$$\frac{\partial L}{\partial \beta} = 0, \quad \frac{\partial L}{\partial y_1} = 0, \quad \frac{\partial L}{\partial y_2} = 0, \quad \frac{\partial L}{\partial y_3} = 0,$$
$$\lambda_j \geqslant 0, \quad \lambda_j g_j = 0, \quad g_j \leqslant 0, \quad j = 1,2,3,4, \tag{10}$$

where L is the Lagrangian function, which is defined as

$$L(\beta, \boldsymbol{\lambda}, \mathbf{y}) = -\beta + \sum_{j=1}^{4} \lambda_j g_j, \tag{11}$$

$\boldsymbol{\lambda}$ are the Lagrangian multipliers and g_j are given by

$$g_j = \Phi[\beta \boldsymbol{\sigma}^E(j) + \mathbf{y}] - \sigma_0^2. \tag{12}$$

The conditions (10) are a nonlinear equation system with eight unknowns β, y_1, y_2, y_3, λ_1, λ_2, λ_3 and λ_4, i.e.

$$\frac{\partial L}{\partial \beta} = -1 + \lambda_1 \frac{\partial g_1}{\partial \beta} + \lambda_2 \frac{\partial g_2}{\partial \beta} + \lambda_3 \frac{\partial g_3}{\partial \beta} + \lambda_4 \frac{\partial g_4}{\partial \beta} = 0, \tag{13}$$

$$\frac{\partial L}{\partial y_1} = \lambda_1 \frac{\partial g_1}{\partial y_1} + \lambda_2 \frac{\partial g_2}{\partial y_1} + \lambda_3 \frac{\partial g_3}{\partial y_1} + \lambda_4 \frac{\partial g_4}{\partial y_1} = 0, \tag{14}$$

$$\frac{\partial L}{\partial y_2} = \lambda_1 \frac{\partial g_1}{\partial y_2} + \lambda_2 \frac{\partial g_2}{\partial y_2} + \lambda_3 \frac{\partial g_3}{\partial y_2} + \lambda_4 \frac{\partial g_4}{\partial y_2} = 0, \tag{15}$$

$$\frac{\partial L}{\partial y_3} = \lambda_1 \frac{\partial g_1}{\partial y_3} + \lambda_2 \frac{\partial g_2}{\partial y_3} + \lambda_3 \frac{\partial g_3}{\partial y_3} + \lambda_4 \frac{\partial g_4}{\partial y_3} = 0, \tag{16}$$

$$\lambda_1 g_1 = \lambda_1 [y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3 - y_1 y_3 - \sigma_0^2] = 0, \tag{17}$$

$$\begin{aligned} \lambda_2 g_2 = \lambda_2 [&(\beta S_{12}+y_1)^2 + (\beta S_{22}+y_2)^2 + (\beta S_{32}+y_3)^2 \\ &- (\beta S_{12}+y_1)(\beta S_{22}+y_2) - (\beta S_{22}+y_2)(\beta S_{32}+y_3) \\ &- (\beta S_{12}+y_1)(\beta S_{32}+y_3) - \sigma_0^2] = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} \lambda_3 g_3 = \lambda_3 [&(\beta S_{13}+y_1)^2 + (\beta S_{23}+y_2)^2 + (\beta S_{33}+y_3)^2 \\ &- (\beta S_{13}+y_1)(\beta S_{23}+y_2) - (\beta S_{23}+y_2)(\beta S_{33}+y_3) \\ &- (\beta S_{13}+y_1)(\beta S_{33}+y_3) - \sigma_0^2] = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} \lambda_4 g_4 = \lambda_4 [&(\beta S_{14}+y_1)^2 + (\beta S_{24}+y_2)^2 + (\beta S_{34}+y_3)^2 \\ &- (\beta S_{14}+y_1)(\beta S_{24}+y_2) - (\beta S_{24}+y_2)(\beta S_{34}+y_3) \\ &- (\beta S_{14}+y_1)(\beta S_{34}+y_3) - \sigma_0^2] = 0. \end{aligned} \tag{20}$$

At first, we assume that yield conditions at the first and third load vertex are not active, this means

$$g_1 < 0 \quad \text{and} \quad g_3 < 0. \tag{21}$$

It will be shown later whether this assumption is true or not. Due to eqns (17) and (19) one has

$$\lambda_1 = \lambda_3 = 0. \tag{22}$$

Equations (13), (14), (15), (16), (18) and (20) remain to be solved for six unknowns, β, y_1, y_2, y_3, λ_2 and λ_4. By doing some modifications of these equations and making use of the program MACSYMA, which is capable of solving some mathematical problems symbolically, we have solved this nonlinear equation system analytically. The shakedown load factor reads

$$\beta = \frac{2\sigma_0}{\sqrt{C_1}} \tag{23}$$

with C_1 being a positive constant depending on the elastic stresses

$$\begin{aligned} C_1 = {} & S_{12}^2 - 2S_{12}S_{14} + S_{14}^2 - S_{12}S_{22} + S_{14}S_{22} + S_{22}^2 + S_{12}S_{24} \\ & - S_{14}S_{24} - 2S_{22}S_{24} + S_{24}^2 - S_{12}S_{32} + S_{14}S_{32} - S_{22}S_{32} + S_{24}S_{32} \\ & + S_{32}^2 + S_{12}S_{34} - S_{14}S_{34} + S_{22}S_{34} - S_{24}S_{34} - 2S_{32}S_{34} + S_{34}^2. \end{aligned} \tag{24}$$

The Lagrangian multipliers λ_2 and λ_4 are

$$\lambda_2 = \lambda_4 = \frac{1}{2\sigma_0\sqrt{C_1}} > 0, \tag{25}$$

and the quantities y_1, y_2 and y_3 are

$$y_1 = -\tfrac{1}{2}\beta(S_{12} + S_{14}), \tag{26}$$

$$y_2 = -\tfrac{1}{2}\beta(S_{22} + S_{24}), \tag{27}$$

$$y_3 = -\tfrac{1}{2}\beta(S_{32} + S_{34}). \tag{28}$$

To check the correctness of the assumption (21), we substitute y_1, y_2 and y_3 in g_1 and g_3, and obtain

$$g_1 = g_3 = \beta^2 C_2 - 4\sigma_0^2 \tag{29}$$

with

$$\begin{aligned} C_2 = {} & S_{12}^2 + 2S_{12}S_{14} + S_{14}^2 - S_{12}S_{22} - S_{14}S_{22} + S_{22}^2 - S_{12}S_{24} \\ & + S_{14}S_{24} + 2S_{22}S_{24} + S_{24}^2 - S_{12}S_{32} - S_{14}S_{32} - S_{22}S_{32} - S_{24}S_{32} \\ & + S_{32}^2 - S_{12}S_{34} - S_{14}S_{34} - S_{22}S_{34} - S_{24}S_{34} + 2S_{32}S_{34} + S_{34}^2. \end{aligned} \tag{30}$$

C_2 is not negative.

By using eqn (23), eqn (29) becomes

$$g_1 = g_3 = \beta^2 C_2 - \beta^2 C_1. \tag{31}$$

For $C_1 > C_2$, one gets

$$g_1 = g_3 < 0,$$

such that the assumption (21) holds.

Until now all conditions in (10) have been satisfied for $C_1 > C_2$. Once elastic stresses for all load vertices are known, the shakedown load factor β follows immediately from eqn (23).

For the case of $C_1 < C_2$, we have solved the equation system analogously. The shakedown load factor reads

$$\beta = \frac{2\sigma_0}{\sqrt{C_2}}, \tag{32}$$

where C_2 is already defined in eqn (30). The Lagrangian multipliers are

$$\lambda_1 = \lambda_3 = \frac{1}{2\sigma_0\sqrt{C_2}} > 0, \tag{33}$$

$$\lambda_2 = \lambda_4 = 0. \tag{34}$$

Quantities y_1, y_2 and y_3 remain unchanged.

For $C_1 = C_2$, the shakedown load factor is

$$\beta = \frac{2\sigma_0}{\sqrt{C_3}} \tag{35}$$

with

$$\begin{aligned} C_3 = {} & S_{12}^2 + S_{14}^2 - S_{12}S_{22} + S_{22}^2 - S_{14}S_{24} + S_{24}^2 \\ & - S_{12}S_{32} - S_{22}S_{32} + S_{32}^2 - S_{14}S_{34} - S_{24}S_{34} + S_{34}^2 \\ > {} & 0. \end{aligned} \tag{36}$$

y_1, y_2 and y_3 are the same as in eqns (26), (27) and (28), and the Lagrangian multipliers are

$$\lambda_1 = \lambda_3 = \lambda_3 = \lambda_4 = \frac{1}{4\sigma_0\sqrt{C_3}} > 0. \tag{37}$$

By putting β, y_1, y_2 and y_3 into g_1, g_2, g_3 and g_4, we obtain

$$g_1 = g_2 = g_3 = g_4 = 0. \tag{38}$$

It means that the yield conditions at all load vertices are active.

Constants C_1, C_2 and C_3 cannot be zero, unless all stress components are hydrostatic, but this case is of no importance.

If the load domain has only two vertices, the solution for the shakedown limit factor is much simpler; it reads

$$\beta = \frac{2\sigma_0}{\sqrt{S_{12}^2+S_{22}^2+S_{32}^2-S_{12}S_{22}-S_{22}S_{32}-S_{12}S_{32}}} = \frac{2\sigma_0}{\sigma_{\text{eff}}}. \tag{39}$$

The denominator of the right-hand side of eqn (39) is the effective stress of the second load vertex. Let σ_{eff} equal the initial yield stress σ_0, then the shakedown load factor is equal to two. This means that in this situation the shakedown limit of a system is twice as large as its elastic limit. This statement is not true for a load domain with more than two vertices.

For the general case that

$$S_{13} \neq S_{12}+S_{14}, \quad S_{23} \neq S_{22}+S_{24}, \quad S_{33} \neq S_{32}+S_{34}, \tag{40}$$

the solutions are more complicated. We give the results without going into mathematical details of the analysis process. There are six possible solutions for the shakedown load factor; they are

$$\beta_1 = \frac{2\sigma_0}{\sqrt{K_1}}, \quad \beta_2 = \frac{2\sigma_0}{\sqrt{K_2}}, \quad \beta_3 = \frac{2\sigma_0}{\sqrt{K_3}} \tag{41}$$

$$\beta_4 = \frac{2\sigma_0}{\sqrt{K_4}}, \quad \beta_5 = \frac{2\sigma_0}{\sqrt{K_5}}, \quad \beta_5 = \frac{2\sigma_0}{\sqrt{K_6}} \tag{42}$$

with K_1, K_2, K_3, K_4, K_5 and K_6 being positive constants which will be given in the Appendix. The correct shakedown load factor of the system is the smallest one of them, i.e.

$$\beta = \min\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}. \tag{43}$$

Now we summarize all steps of the introduced analytical method as follows.

(1) Calculate the elastic stresses for all load vertices.

(2) Locate the position of the point with maximum effective stress.

(3) Compare the principal stresses of this point. If they fulfill the condition (9), one gets the shakedown load factor β by eqn (23), or (32), or (35), depending on the values of C_1 and C_2. Otherwise β follows from eqn (43).

The correctness of the analytical solutions was checked several times and approved by some known analytical solutions as special cases.

4. EXAMPLES

In this section we apply the above analytical method to some examples and then compare the results with those obtained by other methods, so far as available.

Example 1

A square plate with a central hole is subjected to biaxial loadings p_1 and p_2 as shown in Fig. 2. Loadings p_1 and p_2 may vary cyclically between zero and certain maximum magnitudes $\bar{p}_1$, $\bar{p}_2$. For $\bar{p}_1 > \bar{p}_2$, the point A dominates the shakedown behaviour of the system.

The elastic stress components at point A are

$$S_{12} = \sigma_{A_1}, \quad S_{14} = \sigma_{A_2}.$$

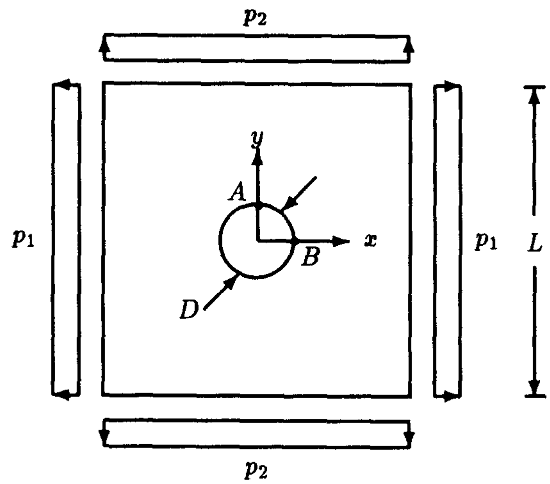

Fig. 2. A square plate with a central hole.

σ_{A_1} and σ_{A_2} stand for elastic stresses at A caused by p_1 and p_2, respectively. The other stress components vanish in this case. Equations (24) and (30) yield

$$C_1 = (\sigma_{A_1} - \sigma_{A_2})^2, \quad C_2 = (\sigma_{A_1} + \sigma_{A_2})^2.$$

Because σ_{A_2} is negative for this example, one has

$$C_1 > C_2.$$

According to eqn (23) the shakedown load factor is

$$\beta = \frac{2\sigma_0}{\sigma_{A_1} - \sigma_{A_2}}.$$

Normally σ_{A_1} and σ_{A_2} must be calculated by using a numerical method, e.g. using FEM or BEM. If the plate is infinite, that means $D/L \to 0$, one gets

$$\sigma_{A_1} = 3p_1 \quad \text{and} \quad \sigma_{A_2} = -p_2.$$

For this case the shakedown diagram is shown in Fig. 3.

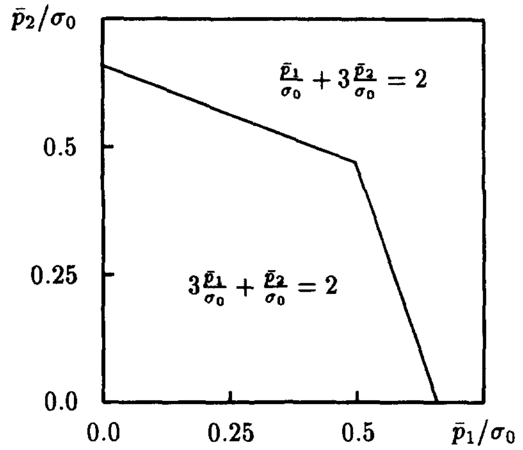

Fig. 3. The shakedown diagram for an infinite plate with a hole.

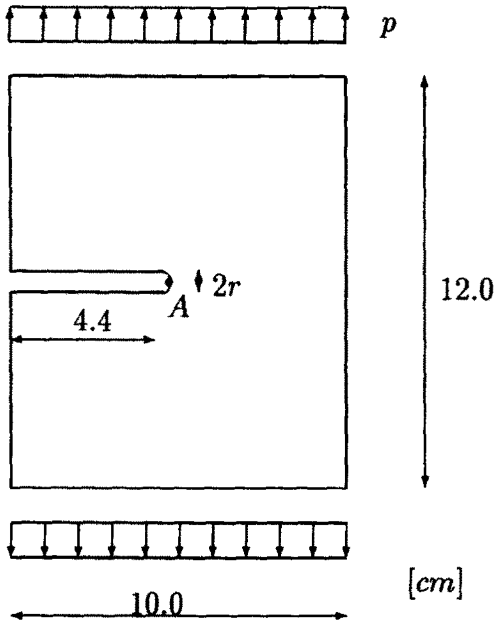

Fig. 4. A compact tension specimen with notch.

Example 2

A compact tension specimen with notch in Fig. 4 is subjected to an uniaxial loading p which varies between 0 and 1 kN. Obviously, the point A at the notch surface dominates the shakedown behaviour of the system.

In this case the shakedown load factor is given simply by

$$\beta = \frac{2\sigma_0}{\sigma_A}$$

with σ_A being the elastic stress at point A, which can be calculated numerically. In this work, however, we use an analytical form suggested by Paris and Sih (1965)

$$\sigma_A = \frac{2K_I}{\sqrt{\pi r}},$$

where K_I is the stress intensity factor calculated from linear fracture mechanics as if the notch would be a crack, and r is the notch root radius. The compact tension specimen with crack has been studied very intensively in fracture mechanics. For the stress intensity factor there exist analytical results [see Murakami (1987)].

Five different values of notch root radius are used to obtain a wide range of shakedown limits. For $\sigma_0 = 24$ kN cm^{-2}, Table 1 shows the results of shakedown load factors, β_a denotes the analytical shakedown load factors and β_n the results obtained by using numerical optimization [see Stein *et al.* (1993a)].

Table 1. The shakedown limits

	β_a	β_n
$r = 0.1$	1.5275	1.6638
$r = 0.2$	1.9410	2.0432
$r = 0.3$	2.1884	2.2201
$r = 0.4$	2.4084	2.4268
$r = 0.5$	2.5380	2.5774

Example 3

A hollow cylinder of inner and outer radii b and a, respectively, is subjected to an internal pressure. The pressure may vary cyclically between 0 and p. For this configuration we have analytical solutions for the elastic stresses

$$\sigma_{rr} = \frac{1}{a^2-b^2}\left\{-pb^2\left(\frac{a^2}{r^2}-1\right)\right\},$$

$$\sigma_{\varphi\varphi} = \frac{1}{a^2-b^2}\left\{pb^2\left(\frac{a^2}{r^2}+1\right)\right\},$$

$$\sigma_z = \nu(\sigma_{rr}+\sigma_{\varphi\varphi}),$$

where ν is the Poisson ratio.

The maximum stresses are located at the inner boundary; they are

$$\sigma^{\mathrm{E}}(1) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \sigma^{\mathrm{E}}(2) = \begin{Bmatrix} -p \\ \dfrac{a^2+b^2}{a^2-b^2}p \\ \nu p\dfrac{2b^2}{a^2-b^2} \end{Bmatrix}.$$

The maximum effective stress is

$$\sigma_{\mathrm{eff}} = \frac{p}{1-(b/a)^2}\sqrt{3+(b/a)^4(1-2\nu)^2}.$$

According to eqn (39) the shakedown load factor is

$$\beta = \frac{2\sigma_0}{\sigma_{\mathrm{eff}}} = \frac{2\sigma_0}{p\sqrt{3+(b/a)^4(1-2\nu)^2}}(1-(b/a)^2),$$

and the maximum pressure p_{s} at which the cylinder will shake down is

$$p_{\mathrm{s}} = \beta p = \frac{2\sigma_0}{\sqrt{3+(b/a)^4(1-2\nu)^2}}(1-(b/a)^2).$$

König (1987) used another method to solve the same problem. In his work the residual stresses are directly constructed. The Tresca linear yielding function is used, and the influence of the axial stress σ_z is neglected. His result for maximum shakedown load is

$$\tilde{p}_{\mathrm{s}} = \sigma_0(1-(b/a)^2).$$

By neglecting the term $(b/a)^4(1-2\nu)^2$, one gets the relation between the maximum shakedown loads of these two methods

$$\frac{\tilde{p}_{\mathrm{s}}}{p_{\mathrm{s}}} = \frac{\sqrt{3}}{2}.$$

5. CONCLUSIONS

A mechanical system consisting of elastic–plastic material subjected to cylically varying loadings behaves in a complicated manner. One of the possible system responses is that the system reaches first an elastic–plastic state, and after a certain number of loading cycles (loading and unloading) the residual stresses become stationary. With the corresponding load domain the system shakes down.

Optimization problems derived from shakedown theory by using finite element discretizations are generally of large dimensions. To treat these problems effectively, special algorithms must be used. Taking the advantage that a system consisting of linear kinematic hardening material fails only locally in the form of alternating plasticity, an analytic method was developed to solve these shakedown problems. So far, the method is valid both for 2-D and 3-D problems, as long as the number of the load domain vertices is not larger than four. In practice the loading domain has usually two or four vertices. Three examples are studied by using the analytical method. There are no difficulties to determine shakedown load factors for other engineering problems with this method.

Though the method is formulated for systems with linear kinematic hardening materials, the results obtained here could be surely references for those with nonlinear hardening materials, as well as for systems with cyclic hardening materials.

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APPENDIX: CONSTANTS IN EQUATIONS (41) AND (42)

The constants K_1, K_2, K_3, K_4, K_5 and K_6 in eqns (41) and (42) are dependent on the elastic stresses. They are defined by

$$K_1 = S_{13}^2 - S_{13}S_{23} + S_{23}^2 - S_{13}S_{33} - S_{23}S_{33} + S_{33}^2, \tag{A.1}$$

$$\begin{aligned} K_2 = {} & 4S_{12}^2 - 4S_{12}S_{13} + S_{13}^2 - 4S_{12}S_{22} + 2S_{13}S_{22} + 4S_{22}^2 + 2S_{12}S_{23} - S_{13}S_{23} - 4S_{22}S_{23} + S_{23}^2 - 4S_{12}S_{32} \\ & + 2S_{13}S_{32} - 4S_{22}S_{32} + 2S_{23}S_{32} + 4S_{32}^2 + 2S_{12}S_{33} - S_{13}S_{33} + 2S_{22}S_{33} - S_{23}S_{33} - 4S_{32}S_{33} + S_{33}^2, \end{aligned} \tag{A.2}$$

$$\begin{aligned} K_3 = {} & S_{13}^2 - 4S_{13}S_{14} + 4S_{14}^2 - S_{13}S_{23} + 2S_{14}S_{23} + S_{23}^2 + 2S_{13}S_{24} - 4S_{14}S_{24} - 4S_{23}S_{24} + 4S_{24}^2 - S_{13}S_{33} \\ & + 2S_{14}S_{33} - S_{23}S_{33} + 2S_{24}S_{33} + S_{33}^2 + 2S_{13}S_{34} - 4S_{14}S_{34} + 2S_{23}S_{34} - 4S_{24}S_{34} - 4S_{33}S_{34} + 4S_{34}^2, \end{aligned} \tag{A.3}$$

$$\begin{aligned} K_4 = {} & S_{12}^2 - 2S_{12}S_{14} + S_{14}^2 - S_{12}S_{22} + S_{14}S_{22} + S_{22}^2 + S_{12}S_{24} - S_{14}S_{24} - 2S_{22}S_{24} + S_{24}^2 - S_{12}S_{32} + S_{14}S_{32} \\ & - S_{22}S_{32} + S_{24}S_{32} + S_{32}^2 + S_{12}S_{34} - S_{14}S_{34} + S_{22}S_{34} - S_{24}S_{34} - 2S_{32}S_{34} + S_{34}^2, \end{aligned} \tag{A.4}$$

$$\begin{aligned} K_5 = {} & S_{12}^2 + 2S_{12}S_{14} + S_{14}^2 - S_{12}S_{22} - S_{14}S_{22} + S_{22}^2 - S_{12}S_{24} - S_{14}S_{24} + 2S_{22}S_{24} + S_{24}^2 - S_{12}S_{32} - S_{14}S_{32} \\ & - S_{22}S_{32} - S_{24}S_{32} + S_{32}^2 - S_{12}S_{34} - S_{14}S_{34} - S_{22}S_{34} - S_{24}S_{34} + 2S_{32}S_{34} + S_{34}^2, \end{aligned} \tag{A.5}$$

$$\begin{aligned} K_6 = {} & S_{12}^2 - 4S_{12}S_{13} + 4S_{13}^2 + 2S_{12}S_{14} - 4S_{13}S_{14} + S_{14}^2 - S_{12}S_{22} + 2S_{13}S_{22} - S_{14}S_{22} + S_{22}^2 + 2S_{12}S_{23} \\ & - 4S_{13}S_{23} + 2S_{14}S_{23} - 4S_{22}S_{23} + 4S_{23}^2 - S_{12}S_{24} + 2S_{13}S_{24} - S_{14}S_{24} + 2S_{22}S_{24} - 4S_{23}S_{24} + S_{24}^2 - S_{12}S_{32} \\ & + 2S_{13}S_{32} - S_{14}S_{32} - S_{22}S_{32} + 2S_{23}S_{32} - S_{24}S_{32} + S_{32}^2 + 2S_{12}S_{33} - 4S_{13}S_{33} + 2S_{14}S_{33} + 2S_{22}S_{33} - 4S_{23}S_{33} \\ & + 2S_{24}S_{33} - 4S_{32}S_{33} + 4S_{33}^2 - S_{12}S_{34} + 2S_{13}S_{34} - S_{14}S_{34} - S_{22}S_{34} + 2S_{23}S_{34} - S_{24}S_{34} + 2S_{32}S_{34} - 4S_{33}S_{34} + S_{34}^2. \end{aligned} \tag{A.6}$$